Bose-Einstein Condensation Nuclear Fusion: Theoretical Predictions and Experimental Tests

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References: Y. E. Kim, Naturwissenschaften (2009) 96: 803-811 (published online14 May 2009) and references therein

Experimental Observations

- (D+D) fusion in free space ($E \geq 10$ keV):

{1} $D + D \rightarrow p + T + 4.03$ MeV

{2} $D + D \rightarrow n + {}^3He + 3.27$ MeV

{3} $D + D \rightarrow {}^4He + \gamma + 23.8$ MeV

$R\{1\} \approx R\{2\}$ and $R\{3\}/R\{1\} \approx 10^{-6}$

- **(D+D) fusion in metal ($E \leq 0.1$ eV) (m represents a host metal lattice or metal particle) :**

{4} $D(m) + D(m) \rightarrow p(m) + T(m) + 4.03$ MeV(m)

{5} $D(m) + D(m) \rightarrow n(m) + {}^3He(m) + 3.27$ MeV(m)

{6} $D(m) + D(m) \rightarrow {}^4He(m) + 23.8$ MeV (m)

Fusion rate R{6} for {6} is much greater than rates R{4} and R{5}

(D+D) fusion in metal (E< 0.1 eV) (m represents a host metal lattice or metal particle) :

{4} $D(m) + D(m) \rightarrow p(m) + T(m) + 4.03\ MeV(m)$

{5} $D(m) + D(m) \rightarrow n(m) + {}^{3}He(m) + 3.27\ MeV(m)$

{6} $D(m) + D(m) \rightarrow {}^{4}He(m) + 23.8\ MeV$

Experimental Observations (as of 2008) (not complete)

From both electrolysis and gas loading experiments

[1] The Coulomb barrier between two deuterons are suppressed

[2] Excess heat production (the amount of exess heat indicates its nuclear origin)

[3] ^{4}He production comensurate with excess heat production, no 23.8 MeV gamma ray

[4] Production of hot spots and micro-scale crators on metal surface

[5] Detection of radiations

[6] Production of nuclear ashes with anomalous rates: R{4} << R {6} and R {5} << R{6}

[7] "Heat-after-death"

[8] Requirement of deuteron mobility (D/Pd > 0.9, electric current, pressure gradient, etc.)

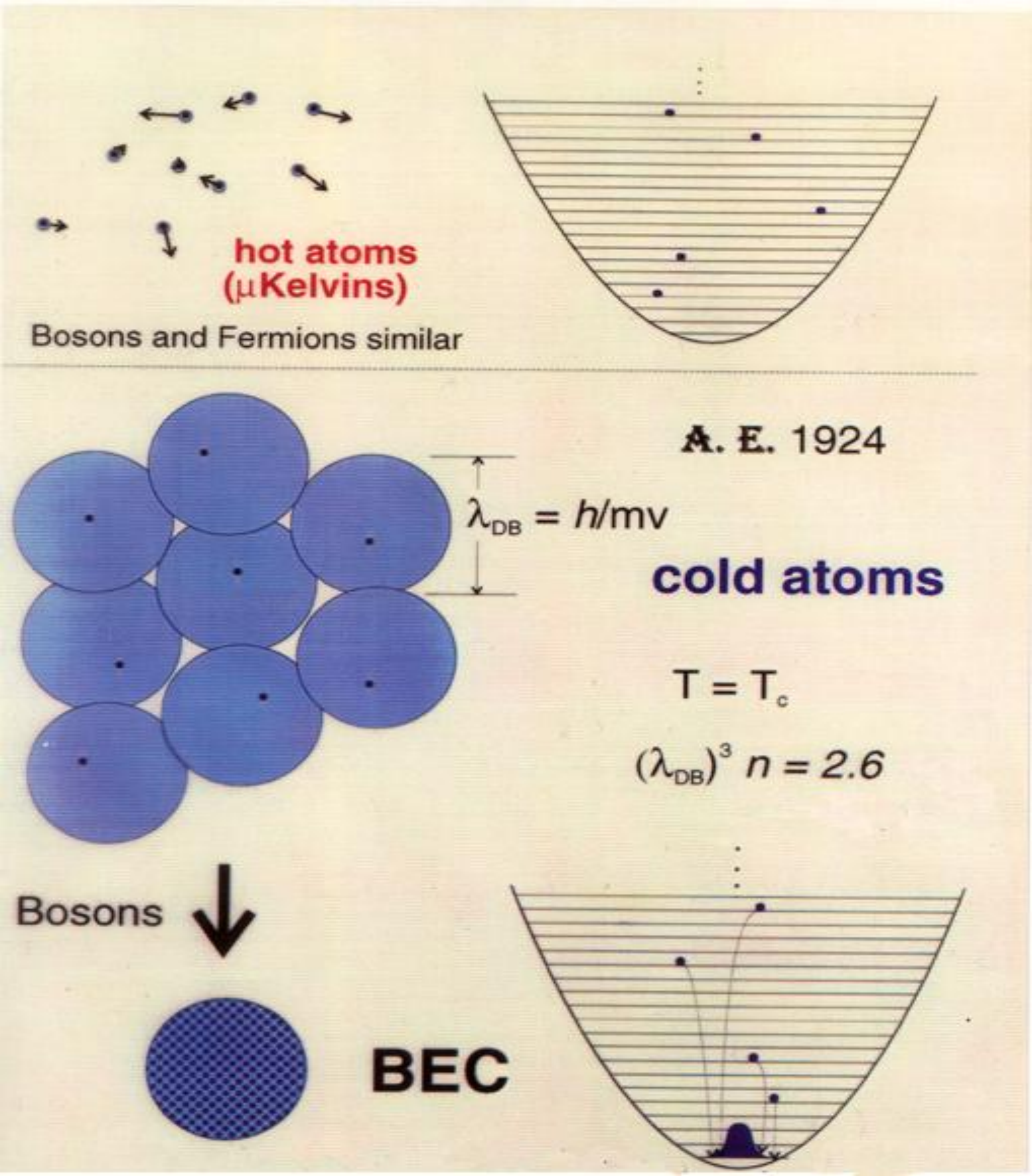
[9] Requirement of deuterium purity (H/D << 1)

[10] More tritium is produced than neutron R(T) >> R(n)

Based on a single physical concept, can we come up with a consistent physical theory which could explain all of the ten experimental observations ?

Deuterons become mobile in metal when electric current (Coehn 1929), and/or pressure gradient is applied !

→Explore a concept of "nuclear" Bose-Einstein Condensation of deuterons in metal for developing a consistent physical theory to explain the experimental observations, [1] through [10].



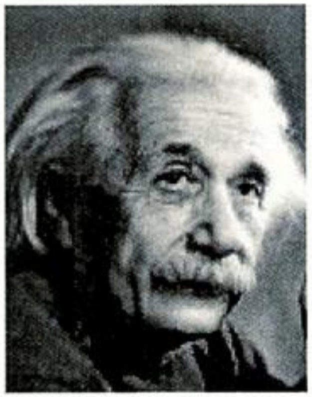
Requirement for Bose-Einstein Condensation (BEC):

$\lambda_{DB} > d$

where d is the average distance between neighboring two Bosons

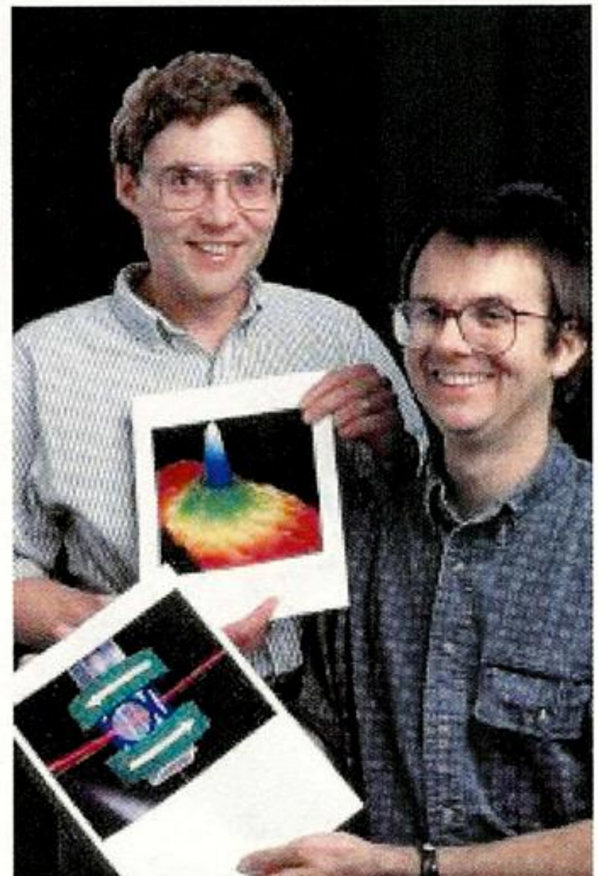
Bose-Einstein Condensation in a gas: a new form of matter at the coldest temperatures in the universe...

Predicted 1924... ...Created 1995



A. Einstein

S. Bose



Created in 1995 by C. Wieman, E. Cornell, W. Ketterle, et al.
→ Nobel Prize in 2000

Atomic BEC vs. Nuclear BEC

BEC Requirement: $\lambda_{DB} > d$, $\lambda_{DB} = \dfrac{h}{m\upsilon}$ or $\upsilon_{kT} < \upsilon_c$

Atomic BEC: $d \approx 7 \times 10^3$ Å = 0.7 μm (for $n_{Rb} = 2.6 \times 10^{12}/cm^3$)

$\upsilon_c \approx 0.6$ cm/sec ($\upsilon_{kT} \approx 0.58$ cm/sec, at $T \approx 170$ n Kelvin)

(~ 2000 atoms in BEC out of ~ 2×10^4 atoms → 10 % in BEC)

(1) Increase λ_{DB} by slowing down neutral atoms using laser cooling and evaporation cooling

Nuclear BEC: $d \approx 2.5$ Å (for $n_D = 6.8 \times 10^{22}/cm^3$ in metal)

$\upsilon_c \approx 0.78 \times 10^5$ cm/sec ($\upsilon_{kT} \approx 1.6 \times 10^5$ cm/sec at T= 300 Kelvin)

(1) Increase λ_{DB} by slowing down charged deuterons using electromagnetic fields, pressure gradient, and/or cooling

(2) Decrease d by compression using ultrahigh pressure device such as Diamond Anvil Cell (DAC)

Fraction F of Deuterons in the BEC State in Metal at Various Temperatures

At 300^{o}K with E_c= 0.00655 eV corresponding to $\lambda_{dB} = \mathbf{d} = 2.5\,\mathring{A}$,

F (E_c) = ~0.084 (8.4% !), (~10% for the atomic BEC case)

Since mobile deuterons in metal are localized within several metal lattice sites, 8.4 % of mobile deuterons with $\upsilon \leq \upsilon_c$ (satisfying $\lambda_{DB} > d$) may not encounter each other frequently enough to form the BEC.

→Need to increase 0.084 (8.4%) to 0.28 (2/7 or 28%) (which is based on a geometrical argument), or

→Collect 8.4% into localized regions by applied EM fields.

At 77.3^{o}K (liquid nitrogen), F(E_c) = ~0.44 (44%)

using Bose-Einstein distribution

At 20.3^{o}K (liquid hydrogen) F(E_c) = ~0.94 (94%)

using Maxwell-Boltzmann distribution.

Boson-Einstein Condensation (BEC) Mechanism

N-**Body Schroedinger Equation for the BEC State**

For simplicity, we assume an isotropic harmonic potential for the deuteron trap.

N-body Schroedinger equation for the system is

$$H\Psi = E\Psi \qquad (1)$$

where Hamiltonian is given by

$$\mathrm{H}=\frac{\hbar^2}{2\mathrm{m}}\sum_{\mathrm{i}=1}^{\mathrm{N}}\Delta_{\mathrm{i}}+\frac{1}{2}\mathrm{m}\omega^2\sum_{\mathrm{i}=1}^{\mathrm{N}}\mathrm{r}_{\mathrm{i}}^2+\sum_{\mathrm{i}<\mathrm{j}}\frac{\mathrm{e}^2}{\left|\mathrm{r_i}-\mathrm{r_j}\right|} \qquad (2)$$

where m is the rest mass of the nucleus.

In presence of electrons, we use the shielded Coulomb potential (Debye screening)

Total Reaction Rate

The total fusion rate R_t is given by

$$R_t = N_{trap} R_{trap} = \frac{N_D}{N} R_{trap} = \frac{1}{4}\left(\frac{3}{\pi}\right)^{1/2} A\Omega V n_D^2 \qquad (19)$$

$$R_{trap} = \frac{1}{2}\left(\frac{3}{\pi}\right)^{3/2} A\Omega \frac{N^2}{D_{trap}^3} = \frac{1}{4}\left(\frac{3}{\pi}\right)^{1/2} A\Omega n_D N$$

where $A = 2S\hbar / (\pi\, me^2)$ with S = 55 keV-barn, D_{trap} is the average diameter of the trap, $D_{trap} = 2 \langle r \rangle$, N_D is the total number of deuterons, N is the number of deuterons in a trap, and n_D is the deuteron density.

Only one unknown parameter is the probability of the BEC ground-state occupation, Ω.

→ **Observation [1] The Coulomb barrier between two deuterons are suppressed**.

⊙ For a single trap (or metal particle) containing N deuterons, we have for primary reactions: leading to secondary reactions

$$\psi_{BEC}\{(N-2)D's+(D+D)\} \rightarrow \psi^{*}\{{}^{4}He+(N-2)D's\} \quad (Q = 23.84\ MeV)$$

⋮

$$\psi_{BEC}\{N D's\} \rightarrow \psi^{*}\{(N/2)\ {}^{4}He\} \quad (Q = (N/2)\ 23.84\ MeV)$$

leading to micro-scale explosions or “melt-down”

where ψ_{BEC} is the Bose-Einstein condensate ground-state (a coherent quantum state) with N deuterons, and ψ^{*} are continuum final states.

⊙Excess energy (Q value) is absorbed by the BEC state and shared by reaction products in the final state.

→ Observation [2] Excess heat production and [3] ^{4}He production, without 23.8MeV gamma rays.

⊙ 3D fusion (D + D + D) and 4 D fusion are possible, but their fusion rates are expected to be much smaller than that of the 2D fusion, leading to secondary effects.

$$\psi_{BEC}\{\text{ND's}\} \rightarrow \psi^*\{(\mathbf{N}/2)\,{}^4\text{He}\} \quad (Q = (\mathbf{N}/2)\ 23.84\ \text{MeV})$$

Conversion of nearly all deuterons to ^{4}He by BECNF in metal grains and particles in the host metal

→ Sustained BECNF and heat production

→ Episodes of "Melt Down" reported by Fleischmann and others

- Excess energies (Q) leading to a micro/nano-scale explosion creating a crater/cavity and a hot spot with firework-like tracks.
- Size of a crater/cavity will depend on number of (D + D) fusions occuring simultaneously in BEC states.
- **→ Observation [4] Production of hot spots and micro-craters.**

$$\psi_{BEC}\left\{(N-2)D's+(D+D)\right\}\rightarrow\psi^{*}\left\{{}^{4}He+(N-2)D's\right\}\quad (Q=23.84\ MeV)$$

leading to secondary reactions

Total Momentum Conservation

- Initial Total Momentum: $\vec{P}_{D^N} \approx 0$
- Final Total Momentum:

$$\{6\}\quad \vec{P}_{D^{N-2}\,{}^4He} \approx 0, \qquad \langle T_D \rangle \approx \langle T_{{}^4He} \rangle \approx \frac{Q\{6\}}{N}$$

- <T> is the average kinetic energy.

- ~1 keV (up to 23.8 MeV) deuterons from {6} lose energies by electrons and induce X-rays, γ-rays, and Bremsstrahlung X-rays.

→ **Observation [5] Detection of radiations.**

Selection Rule for Two-Species Case $\frac{Z_1}{m_1} = \frac{Z_2}{m_2}$

(m is mass number approximately given in units of the nucleon mass)

$$\frac{Z_1(D)}{m_1(D)} = \frac{1}{2}, \quad \left(\frac{Z_2(p)}{m_2(p)} = 1, \quad \frac{Z_2(T)}{m_2(T)} = \frac{1}{3}\right), \quad \left(\frac{Z_2(n)}{m_2(n)} = 0, \quad \frac{Z_2(^3He)}{m_2(^3He)} = \frac{2}{3}\right)$$

•Reactions {4} and {5} are forbidden/suppressed→ reaction rates are small

{4} D(m) + D(m) → p(m) + T(m) + 4.03 MeV (m)

{5} D(m)+D(m) → n(m) + ^{3}He(m) + 3.27 MeV (m)

$$\frac{Z_1}{m_1}\frac{(D)}{(D)} = \frac{Z_2(^4He)}{m_2(^4He)} = \frac{1}{2}$$

•Reaction {6} is allowed → reaction rate is large

{6} D(m)+D(m)→^{4}He(m) + 23.8 MeV (m)

→ This explains Observation [6] R(4) << R(6) and R(5) << R(6).

- **Heat after Death (Observation [7])**

  Because of mobility of deuterons in Pd nanoparticle traps, a system of ~10^{22} deuterons contained in ~ 10^{18} Pd nanoparticle traps is a dynamical system (in 3 g of 5 nm Pd nanoparticles).

  BEC states are continuously attained in a small fraction of the ~10^{18} Pd particle traps and undergo BEC fusion processes, until the formation of the BEC state ceases.

- **Deuteron Mobility Requirement (Observation [8])**

  D/Pd $\geq$ ~0.9 is required for sustaining deuteron mobility in Pd.

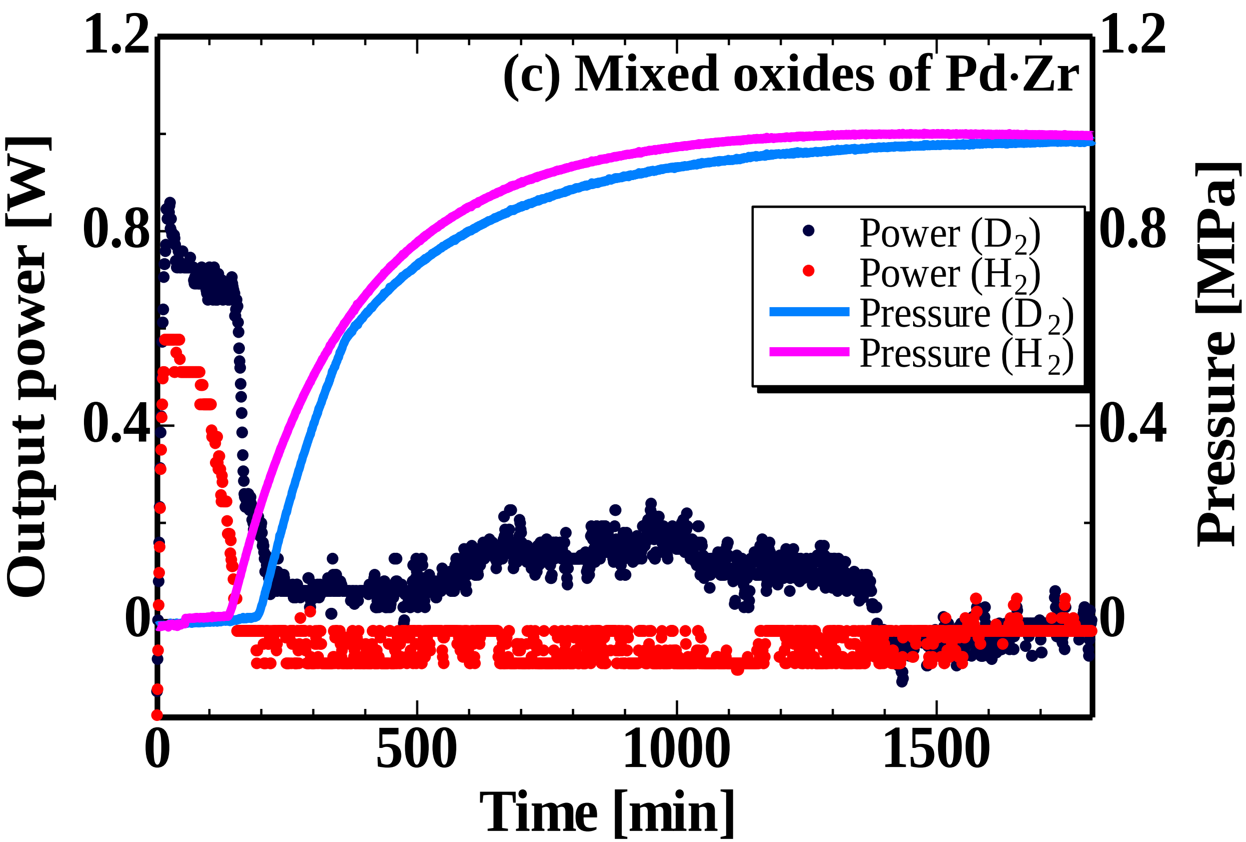
  Electric current or **pressure gradient** is required.

- **Deuterium Purity Requirement (Observation [9])**

  Because of violation of the two-species selection rule,

  presence of hydrogens in deuteriums will surpress the formation of the BEC states, thus diminishing the fusion rate due to the BEC mechanism.

Fig. 3(c): A. Kitamura *et al*., Physics Letters A, 373 (2009) 3109-3112.



- **Consistent with [8] the requirement of deuteron mobility (D/Pd > 0.9, electric current, pressure gradient, etc.)**

- **Output power of 0.15 W corresponds to $R_t \approx$ 1 x 10^9 DD fusions/sec for D+D → ^{4}He + 23.8 MeV**

BEC Mechanism on Reactions {4} and {5} (Secondary Reactions)

- **R{4} << R{6}, R{5} << R{6}, due the selection rule**

$$\{4\} \quad D(m) + D(m) \rightarrow p(m) + {}^{3}He(m) + 4.03MeV(m)$$

$$\{5\} \quad D(m) + D(m) \rightarrow n(m) + T(m) + 3.27MeV(m)$$

$$\{6\} \quad D(m) + D(m) \rightarrow {}^{4}He(m) + 23.8MeV(m)$$

where neutron, n(m), is at energies ~keV.

- **~keV neutron(m) from Reaction {5} can undergo further reactions, {12}, and/or {13} below:**

{12} n(m) + D(m) (in BEC State) → T(m) +6.26 MeV(m)

{13} n(m) + D → T + γ + 6.26 MeV

→ Reactions {12} and {13} produce more tritiums than neutrons, R(T) > R(n).

→ R(T) > R(^{3}He)

→This explains Observation [10] more tritium is produced than neutron.

Experimental Tests of Predictions of BECNF theory

1. Tests based on the average size of metal particles
2. Tests for reaction-rate increases by applied EM fields
3. Tests for resistivity change
4. Tests for scalability

Basic Fundamental Tests of "Nuclear" Bose-Einstein Condensation of Deuterons in Metal

5. Ultrahigh pressure experimental tests
6. Low temperature experimental tests

→ These experimental tests are needed

(1) to improve and/or refine the theory, and also

(2) to achieve 100 % reproducibility for experimental results, and for possible practical applications.

Experimental Tests

1. Tests based on the average size of metal nanoparticles

- The total fusion rate is given by

$$R_t = N_{trap} R_{trap} = \frac{N_D}{N} R_{trap} = \frac{1}{4}\left(\frac{3}{\pi}\right)^{1/2} A\Omega V n_D^2 \qquad (19)$$

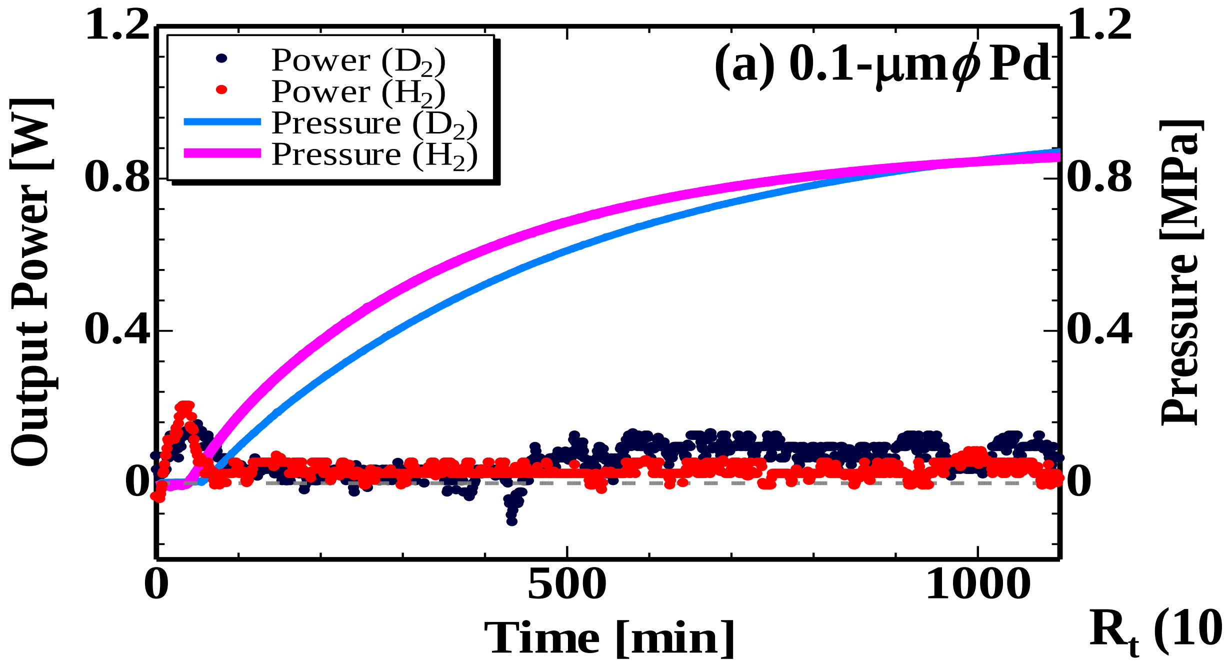
where N_{trap} is the total number of traps, N_D is the total number of deuterons, N is the number of deuterons in a trap, and Ω is the probability of the BEC state occupation.

For the case of Ω proportional to the ratio of surface area/ volume of each particle: $\Omega \propto N^{-1/3}$ or $\Omega \propto D_{trap}^{-1}$

$$R_t(D_{trap}) \propto \frac{1}{D_{trap}} \qquad \frac{R_t(5nm)}{R_t(10nm)} \approx 2, \quad \frac{R_t(2nm)}{R_t(10nm)} \approx 5, \textbf{ etc.}$$

R_t (smaller Pd particles) > R_t (larger Pd particles)

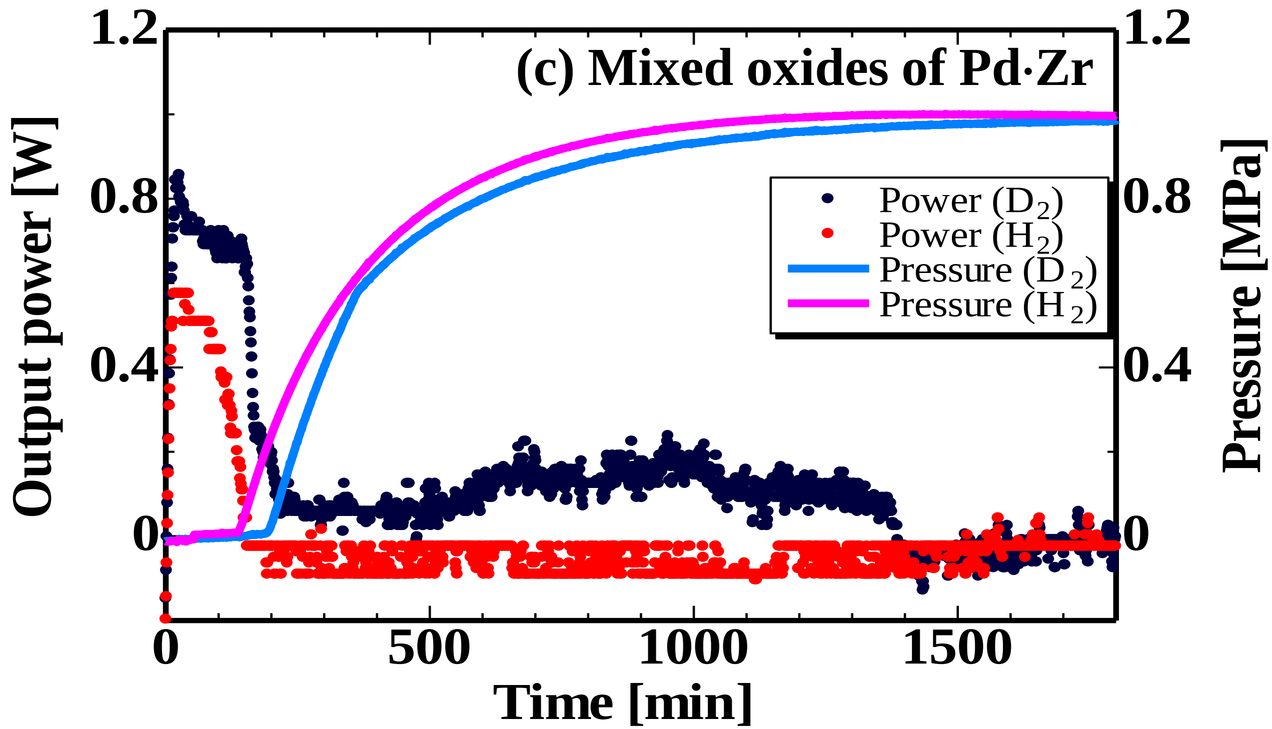
- The above theoretical prediction [Kim, Naturwissenschaften 96 (2009) 803-811 (14 May 2009)] is experimentally confirmed by A. Kitamura et al./ Physics Letters A **373** (2009) 3109-3112 (4 July 2009)



A. Kitamura et al./
Physics Letters A 373
(2009) 3109-3112

5 g of
100 nm Pd particles

R_t (10.7-nmφPd) > R_t (0.1μmφPd)



3 g of
10.7 nm Pd particles

$$\frac{\mathbf{R_t}\left(\mathbf{10.7\ nm\ Pd}\right)}{\mathbf{R_t}\left(\mathbf{100\ nm\ PD}\right)} = \left(\frac{\mathbf{100\ nm}}{\mathbf{10.7\ nm}}\right) \approx \mathbf{9.3}$$

2. Tests for reaction-rate increases by applied EM fields

Increase of reaction-rate is expected by increase of BEC deuteron fraction which can be accomplished by applied EM fileds (electric currents (AC or DC), external electric field, external magnetic field, etc.)

3. Tests for resistivity change

Measure resistivity change which is expected when BEC occurs.

4. Tests for scalability

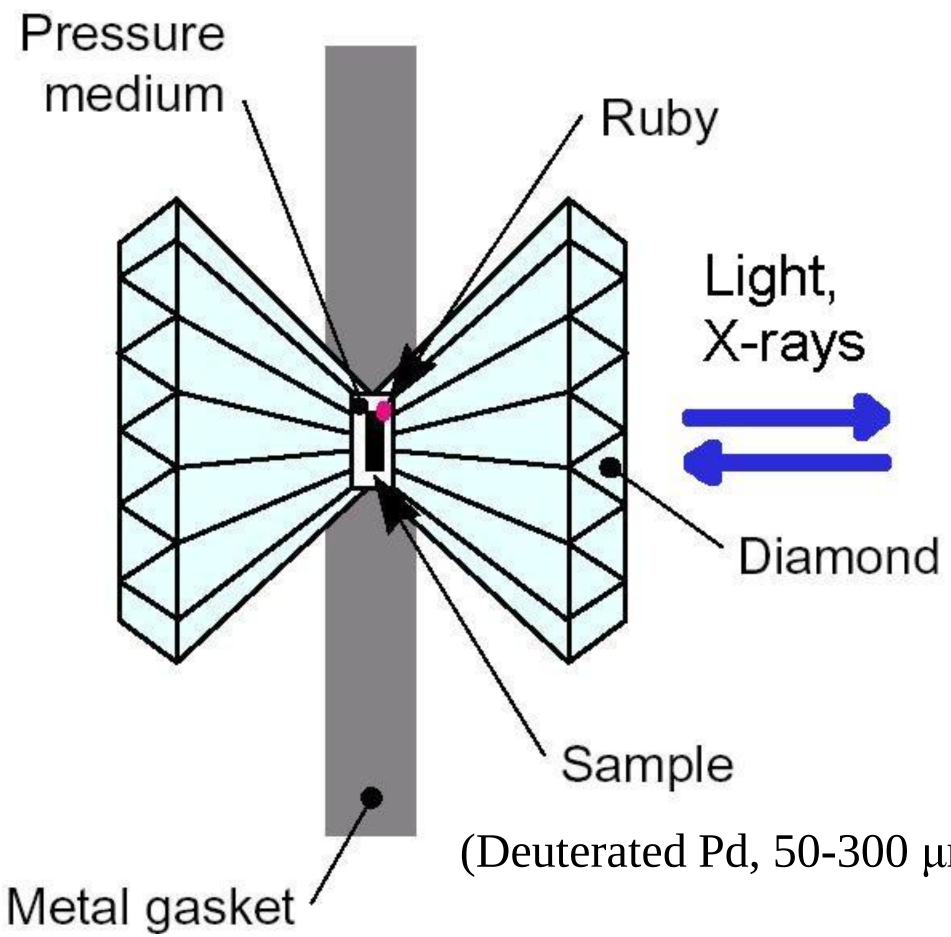
$$R_t \propto N_{trap} R_{trap} \propto N_{trap} \text{ for the same } R_{trap}$$

$$\rightarrow \quad \frac{R_t(\text{30g Pd particles})}{R_t(\text{3g Pd particles})} = 10, \text{etc.}$$

$$\frac{Q\{5\}}{N} \approx \quad \sim keV$$

5. Proposed Basic Fundamental Tests of “Nuclear” BEC - I

⊙ Ultrahigh pressure experimental tests



(Deuterated Pd, 50-300 μm)

Schematics of the core of a diamond anvil cell.
The diamond size is a few millimeters at most

- Apply electric current through the sample.

- Sudden change in the resistivity is expected when deuterons form a BEC state at some pressure.

- Emission of radiations and neutrons may be expected when BECNF occurs.

- Possibility of using laser beam to measure Raman scattering frequency shifts.

6. Proposed Basic Fundamental Tests of "Nuclear" BEC - II

- Low temperature experiments
- **Test of theoretical prediction $R_t(T_{low}) > R_t(T_{high})$**
- **At liquid nitrogen temperature (77.3 Kelvin), the fraction $F(E_c)$ of mobile deuterons in metal satisfying $\lambda_{DB} > d \approx 2.5$ Å or $E < E_c = 0.00655$ eV is $F(E_c) = \sim 0.44$ (~44% !)**
- **At liquid hydrogen temperature (20.3 Kelvin), $F(E_c) = \sim 0.94$ (~94 % !), using Maxwell-Boltzmann distribution.**
- Apply electric current through the sample.
- Change in the resistivity is expected when deuterons form a BEC state at some lower temperatures
- Emission of radiations and neutrons may be expected when BECNF occurs, as secondary effects.
- Shifts in Raman scattering frequencies are expected when BEC occurs

Other Potential Applications of the Concept of Bose-Einstein Condensation of Deuterons in Metal

1. Transmutation
2. Transient Acoustic Cavitation Fusion
3. High Temperature Superconductivity of metal/alloy hydrides/deuterides

Conclusions and Summary

- **BECNF Theory provides a consistent conventional theoretical description of the experimental observations, [1] through [10].**

- **Experimental tests of a set of six (6) key theoretical predictions are proposed including two basic fundamental experimental tests of the concept of "nuclear" Bose-Einstein condensation of deuterons in metal**

- **Experimental tests of the predictions of the BECNF theory are needed in order (1) to improve and/or refine the theory, and also (2) to achieve 100 % reproducibility for practical applications.**

- **If the theoretical predictions are all confirmed experimentally, the concept of Bose-Einstein condensation of deuterons in metal may become a new discovery.**

Backup Slides

Example with 3g of 50 Å Pd particles

- Total number of Pd atoms in 3g, N_{Pd} = 1.7 x 10^{22} Pd atoms
  $N_{Pd} = 3g \times (6.02 \times 10^{23})/106.4g \approx 1.7 \times 10^{22}$ Pd atoms
  For $N_D \approx N_{Pd}$, $N_D = 1.7 \times 10^{22}$ D atoms

- The number density of Pd, $n_{Pd} \cong 6.8 \times 10^{22} cm^{-3} = n_D$

  n_{Pd} = 12.03 g $cm^{-3} \times (6.02 \times 10^{23})/106..4$ g $\approx 6.8 \times 10^{22} cm^{-3}$

- One Pd particle of diameter ~50 Å contains $N = n_D \left(\frac{\pi}{6}\right)\left(50\,\text{Å}\right)^3 \approx 4450$ deuterons

- In 3g of Pd particles, the total number of Pd particle traps is

$$\frac{N_D}{N} = \frac{N_{Pd}}{N} = \frac{1.7 \times 10^{22}}{4.45 \times 10^3} \approx 3.8 \times 10^{18}$$

particle traps

- For comparison, ~ 2000 atoms are trapped for the atomic case.

Theoretical derivations of BEC nuclear fusion rates are given in the following references:

Approximate Solution of Many-Body Schroedinger Equation

1. Y.E. Kim and A.L. Zubarev, “Equivalent Linear Two-Body Method for Many-Body Problems", J. Phys. B: At. Mol. Opt. Phys. **33**, 55 (2000).
2. Y.E. Kim and A.L. Zubarev, “Equivalent Linear Two-Body Method for Bose-Einstein Condensates in Time-Dependent Harmonic Traps", Physical Review **A66**, 05362 (2002).

Optical Theorem Formulation of Nuclear Reactions

3. Y.E. Kim, Y.J. Kim, A.L. Zubarev, and J.-H. Yoon, “Optical Theorem Formulation of Low-Energy Nuclear Reactions", Physical Review **C55**, 801 (1997).

Nuclear Fusion Rates for Deuterons in the BEC State

4. Y.E. Kim and A.L. Zubarev, “Nuclear Fusion for Bose Nuclei Confined in Ion Traps", Fusion Technology **37**, 151 (2000).
5. Y.E. Kim and A.L. Zubarev, “Ultra Low-Energy Nuclear Fusion of Bose Nuclei in Nano-Scale Ion Traps", Italian Physical Society Conference Proceedings, Vol. **70**, (May 2000), pp. 375-384.

Fusion Reaction Rates

Our final theoretical formula for the nuclear fusion rate R_{trap} for a single trap containing N deuterons is given by

$$\mathbf{R_{trap}} = \Omega\mathbf{BN}\omega^2 \qquad (18)$$

$$\omega^2 = \sqrt{\frac{3}{4\pi}}\,\alpha\left(\frac{\hbar\mathbf{c}}{\mathbf{m}}\right)\frac{\mathbf{N}}{\langle\mathbf{r}\rangle^3}$$

where $\langle r\rangle$ is the radius of trap/atomic cluster, $\langle r\rangle=\langle\Psi|r|\Psi\rangle$,

B is given by $B=3Am/(8\pi\alpha\hbar c)$,

N is the average number of Bose nuclei in a trap/cluster.

A is given by $A=2Sr_B/(\pi\hbar)$, where $r_B=\hbar^2/(2\mu e^2)$, $\mu=m/2$,

S is the S-factor for the nuclear fusion reaction between two deuterons (for D(D,p)T and D(d,n)^{3}He reactions, S $\approx$ 55 keV-barn)

All constants are known except Ω, which is the probability of the BEC ground state occupation

Equivalent Linear Two-Body (ELTB) Method

(Kim and Zubarev, Physical Review A **66**, 053602 (2002))

For the ground-state wave function Ψ, we use the following approximation

$$\Psi(\vec{r},\ldots\vec{r}_N) \approx \tilde{\Psi}(\rho) = \frac{\Phi(\rho)}{\rho^{(3N-1)/2}} \qquad (3)$$

where $\rho = \left[\sum_{i=1}^{N} r_i^2\right]^{1/2}$

It has been shown that approximation (3) yields good results for the case of large N (Kim and Zubarev, J. Phys. B: At. Mol. Opt. Phys. **33**, 55 (2000))

By requiring that Ψ must satisfy a variational principle $\delta\int \Psi^* H \Psi d\tau = 0$ with a subsidiary condition $\int \Psi^* \Psi d\tau = 1$, we obtain the following Schrödinger equation for the ground state wave function Φ(ρ)

$$[-\frac{\hbar^2}{2m}\frac{d^2}{d\rho^2} + \frac{m}{2}\omega^2\rho^2 + \frac{\hbar^2}{2m}\frac{(3N-1)(3N-3)}{4\rho^2} + V(\rho)]\Phi = E\Phi \qquad (4)$$

where $V(\rho) = \dfrac{2N\Gamma(3N/2)}{3\sqrt{2\pi}\Gamma(3N/2-3/2)\rho}$ (5)

Optical Theorem Formulation of Nuclear Fusion Reactions (Kim, et al. Physical Review C 55, 801 (1997))

In order to parameterize the short-range nuclear force, we use the optical theorem formulation of nuclear fusion reactions. The total elastic nucleus-nucleus amplitude can be written as

$$f(\theta) = f^{c}(\theta) + \tilde{f}(\theta) \tag{6}$$

where $f^{c}(\theta)$ is the Coulomb amplitude, and $\tilde{f}(\theta)$ can be expanded in partial waves

$$\tilde{f}(\theta) = \sum_{l} (2l+1) e^{2i\delta_l^c} f_l^{n(el)} P_l(\cos\theta) \tag{7}$$

In Eq. (7), δ_l^c is the Coulomb phase shift, $f_l^{n(el)} = (S_l^n - 1)/2ik$, and s_l^n is the *l*-th partial wave S-matrix for the nuclear part.

For low energy, we can write (optical theorem)

$$\mathrm{Im}\, f_l^{n(el)} \approx \frac{k}{4\pi} \sigma_l^r \tag{8}$$

where σ_l^r is the partial wave reaction cross section.

In terms of the partial wave t-matrix, the elastic scattering amplitude, $f_l^{n(el)}$ can be written as

$$f_l^{n(el)} = -\frac{2\mu}{\hbar^2 k^2} < \psi_l^c | t_l | \psi_l^c > \tag{9}$$

where ψ_l^c is the Coulomb wave function.

Parameterization of the Short-Range Nuclear Force

For the dominant contribution of only *s*-wave, we have

$$\mathrm{Im}\, f_0^{n(el)} \approx \frac{k}{4\pi}\sigma^r \tag{10}$$

and

$$f_0^{n(el)} = -\frac{2\mu}{\hbar^2 k^2}\langle \psi_0^c | t_0 | \psi_0^c \rangle \tag{11}$$

Where σ^r is conventionally parameterized as

$$\sigma^r = \frac{S}{E} e^{-2\pi\eta} \tag{12}$$

$\eta = \frac{1}{2kr_B}, r_B = \frac{\hbar^2}{2\mu e^2}, \mu = m/2$, $e^{-2\pi\eta}$ is the "Gamow" factor,

and *S* is the *S*- factor for the nuclear fusion reaction between two nuclei.

From the above relations, Eqs. (10), (11), and (12), we have

$$\frac{k}{4\pi}\sigma^r = -\frac{2\mu}{\hbar^2 k^2} < \psi_0^c | \mathrm{Im}\, t_0 | \psi_0^c > \tag{13}$$

For the case of *N* Bose nuclei, to account for a short range nuclear force between two nuclei, we introduce the following Fermi pseudo-potential $V^F(\vec{r})$

$$\mathrm{Im}\, t_0 = \mathrm{Im} V^F(\vec{r}) = -\frac{A\hbar}{2}\delta(\vec{r}) \tag{14}$$

where the short-range nuclear-force constant *A* is determined from Eqs. (12) and (13) to be $A = 2Sr_B / \pi\hbar$.

For deuteron-deuteron (DD) fusion via reactions *D(d,p)T* and *D(d,n)*^{3}*He,* the *S*-factor is *S* = 110 *KeV-barn.*

Derivation of Fusion Probability and Rates

For N identical Bose nuclei confined in an ion trap, the nucleus-nucleus fusion rate is determined from the trapped ground state wave function ψ as

$$R_t = -\frac{2}{\hbar}\frac{\Sigma_{i<j} < \psi|\mathrm{Im}\, t_{ij}|\psi >}{< \psi|\psi >} \tag{15}$$

where $\mathrm{Im}\, t_{ij}$ is given by the Fermi potential Eq. (14), $\mathrm{Im}\, t_{ij} = -A\hbar\delta(\vec{r})/2$.

From Eq. (15), we obtain for a single trap

$$R_t = \sqrt{\frac{3}{4\pi}}\Omega B\alpha\left(\frac{\hbar c}{m}\right)Nn_B \tag{16}$$

where Ω is the probability of the ground state occupation, $\alpha = e^2/\hbar c, n_B = N/<r>^3$ is Bose nuclei density in a trap, and $B = 3Am/8\pi c$ with $A = 2Sr_B/\pi\hbar$

For the case of multiple ion traps (atomic clusters or bubbles), the total ion-trap nuclear fusion rate *R* per unit time and per unit volume, can be written as

$$R = n_t\sqrt{\frac{3}{4\pi}}\Omega B\alpha\left(\frac{\hbar c}{m}\right)Nn_B \tag{17}$$

where n_t is a trap number density (number of traps per unit volume) and *N* is the average number of Bose nuclei in a trap.

Selection Rule for Two Species Case

Mean-Field Theory of A Quantum Many-Particle System

(Hartree-Fock Theory)

We consider a mixture of two different species of positively charged bosons, with N_1 and N_2 particles, charges $Z_1 \geq 0$ and $Z_2 \geq 0$, and rest masses m_1 and m_2, respectively. We assume $\mathbf{V_i(\vec{r}) = m_i \omega_i^2 r^2 / 2.}$

The mean-field energy functional for the two-component system is given by generalization of the one-component case

$$E = \sum_{i=1}^{2} E_i + E_{\text{int}}, \qquad (20)$$

where

$$E_i = \int d\vec{r} \, \frac{\hbar^2}{2m_i} \left| \nabla \psi_i \right|^2,$$

$$E_{\text{int}} = \frac{e^2}{2} \int d\vec{x} d\vec{y} \, \frac{\left(Z_1 n_1(\vec{x}) + Z_2 n_2(\vec{x})\right)\left(Z_1 n_1(\vec{y}) + Z_2 n_2(\vec{y})\right)}{\left|\vec{x} - \vec{y}\right|}$$

$n_i = \left|\psi_i\right|^2$, is density of specie i, and $\int d\vec{r} n_i(\vec{r}) = N_i.$ (21)

The minimization of the energy functional, Eq. (20), with subsidiary conditions, Eq. (21), leads to the following time-independent mean-field equations.

$$-\frac{\hbar^2}{2m_i}\nabla^2\psi_i(\vec{r})+(V_i+W_i)\psi_i(\vec{r})=\mu_i\psi_i(\vec{r}), \qquad (22)$$

where

$$W_i(\vec{r})=e^2\int d\vec{y}\left[Z_i^2 n_i^2(\vec{y})+Z_1Z_2n_1(\vec{y})n_2(\vec{y})\right]/\left(|\vec{r}-\vec{y}|n_i(\vec{y})\right), \qquad (23)$$

and μ_i are the chemical potentials, $\mu_i=\frac{\partial E}{\partial N_i}$. **(general thermodynamics identity).**

In the Thomas-Fermi (TF) approximation (neglects the kinetic energy terms in Eq. (22)), Eq. (22) reduce to

$$\mu_i=V_i+W_i \qquad (24)$$

which leads to the selection rule (derivation in a backup slide), $\frac{\mathbf{Z_1}}{\mathbf{m_1}}=\frac{\mathbf{Z_2}}{\mathbf{m_2}}$

Selection Rule

For the BEC mechanism for LENR, we obtain nuclear charge-mass selection rule (approximate).

Nuclear mass-charge selection rule: $\mu_i = V_i + W_i$

We can obtain from Eq. (24) that

$$\mu_2 - \frac{Z_2}{Z_1}\mu_1 = \left(\frac{m_2\omega_2^2}{m_1\omega_1^2} - \frac{Z_2}{Z_1}\right)\frac{m_1\omega_1^2}{2}r^2 .$$

Since μ_i are independent of r, we have proved that Eq. (23) has non-trivial solution if and only if

$$\left(\frac{m_2\omega_2^2}{m_1\omega_1^2} - \frac{Z_2}{Z_1}\right) = 0, \quad \text{or} \quad \lambda = \frac{m_2\omega_2^2 Z_1}{m_1\omega_1^2 Z_2} = 1. \qquad (25)$$

If we assume $\omega_1=\omega_2$, $E_1(\text{G.S.}) = \frac{3\hbar\omega_1}{2} = E_2(\text{G.S.}) = \frac{3\hbar\omega_2}{2}$,

and we have from Eq. (25), $\lambda=m_2Z_1/m_1Z_2=1$ or

$$\frac{Z_1}{m_1} = \frac{Z_2}{m_2} \qquad (26)$$

Fraction of Deuterons in the BEC State in Metal at Room Temperature

For Bose-Einstein distribution, the distribution function is given by

$$\mathbf{n_{BE}}(\mathbf{E}) = \frac{1}{\mathbf{e^{\alpha}e^{E/kT}} - 1}$$

where $\mathbf{e^{\alpha}} \approx \left(2\pi\mathbf{mkT}\right)^{3/2} / \mathbf{n_D h}^3$ with the deuteron density, $n_{D.}$

Using the density of (quantum) states N(E) given by

$$\mathbf{N(E)dE} = \frac{\mathbf{4\pi V}}{\mathbf{h}^3}\mathbf{(2m^3)^{1/2}}\sqrt{\mathbf{E}}\,\mathbf{dE}$$

the total number of **N** can be calculated as

$$\mathbf{N} = \int_0^{\infty} \mathbf{n_{BE}(E)N(E)dE} = \int_0^{\infty} \frac{\mathbf{N(E)dE}}{\mathbf{e^{\alpha}e^{E/kT}} - 1}$$

A fraction $F(E_c)$ of **N** deuterons below the critical energy E_c satisfying $\lambda_{dB} = \mathbf{d}\left(\lambda_{dB} \equiv \mathbf{h}/\mathbf{m}\upsilon\right)$ can be calculated as

$$\mathbf{F(E_c)} = \frac{1}{\mathbf{N}}\int_0^{E_c} \mathbf{n_{BE}(E)N(E)dE}$$

For E_c= 0.00655 eV corresponding to $\lambda_{dB} = \mathbf{d} = 2.5\,\mathring{A}$,
F (E_c) = 0.084 (8.4% !), compared to ~10% for the atomic BEC case.