Sub-barrier fusion and selective resonant tunneling

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The cross section of deuteron-triton sub-barrier fusion is calculated using the selective resonant tunneling model with the assumption of a square-well nuclear potential. A complex potential is assumed to describe the absorption inside the nuclear well. The surprisingly good agreement between the theoretical calculation and the experimental data implies that the compound nucleus model might not be applicable to the light-nuclei sub-barrier fusion. Instead, the selective resonant tunneling model is proposed.

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I. INTRODUCTION

For more than 40 years, controlled nuclear fusion research has been concentrated on deuteron-triton fusion because their fusion cross section is greater than that of deuteron-deuteron fusion by a factor of several hundreds, although the Coulomb barrier for $d+t$ is almost the same as that for $d+d$. The resonance of the $d+t$ state near 100 keV is considered as the reason for such a large cross section [1]. A simple square-well model is applied to describe this $d+t$ nuclear interaction, and an imaginary part of the potential is introduced to describe this fusion reaction [2,3]. It is interesting to notice that while the real part of the potential is mainly derived from this resonance energy, the imaginary part of the potential is determined by the Gamow factor at the energy of this resonance. The good agreement between the experimental data and the quantum-mechanics calculation suggests a selective resonant tunneling model [4]. It is different from the conventional compound nucleus model, because the penetrating particle will keep its memory of the phase factor of its wave function. The implication of this selective resonant tunneling model is further explored for the light nuclei fusion.

II. MATCHING DAMPING

When a deuteron is injected to a triton, their relative motion can be described by a reduced radial wave function $\phi(r)$, which is related to the solution of the Schrödinger equation, $\Psi(r,t)$, by

$$\Psi(r,t)=\frac{1}{\sqrt{4\pi r}}\phi(r)\exp\left(-i\frac{E}{\hbar}t\right). \tag{1}$$

The Hamiltonian has an isotropic potential (Fig. 1) which is composed of a square well $(r<a)$, and a Coulomb potential $(r>a)$. Nuclear interaction would introduce a phase shift δ in the wave function; then, the cross section of the reaction σ_r may be related to this phase shift as [5]

$$\sigma_r^{(o)}=\frac{\pi}{k^2}(1-|e^{i\delta_o}|^2). \tag{2}$$

The subscript and the superscript ''o'' denote that only the S wave is considered. Here, k is the wave number for the relative motion. When the nuclear potential is a complex potential, the phase shift δ_o is a complex number also. It is convenient to assume

$$\cot(\delta_o)=W_r+iW_i. \tag{3}$$

Then

$$\sigma_r^{(o)}=\frac{\pi}{k^2}\left[\frac{-4W_i}{W_r^2+(W_i-1)^2}\right]. \tag{4}$$

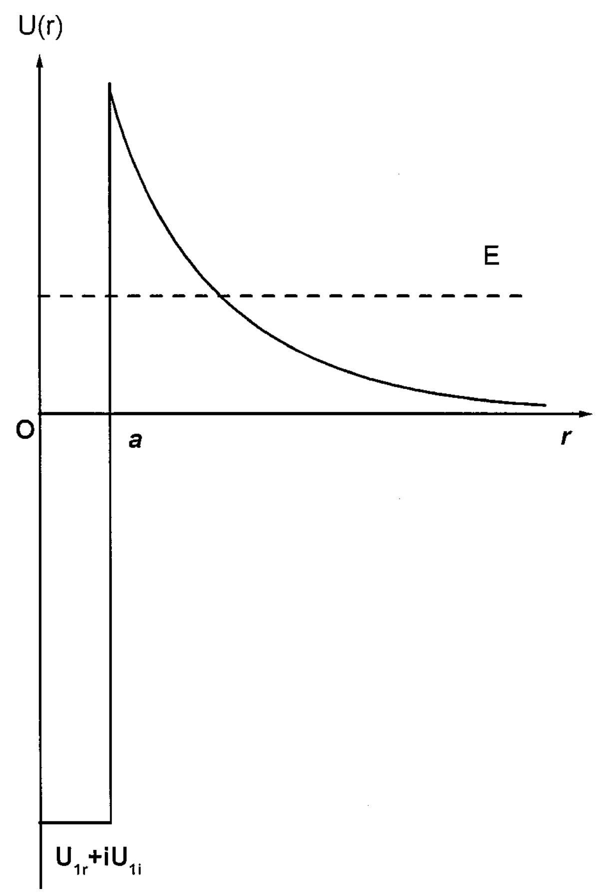

FIG. 1. Schematics for square-well nuclear potential and Coulomb barrier.

$\sigma_r^{(o)}$ reaches its maximum when

$$\begin{cases} W_r = 0, \\ W_i = -1. \end{cases} \tag{5}$$

It is evident that $W_r = 0$ corresponds to the condition for resonance, i.e.,

$$\mathrm{Re}(\delta_o) = \frac{n\pi}{2} \quad (n \text{ is an odd integer}). \tag{6}$$

On the other hand, W_i is related to the imaginary part of the nuclear potential, U_{1i}. When $U_{1i} = 0, W_i = 0$. It simply means that the fusion cross section is zero, if there is no absorption. However, if $U_{1i} \to -\infty$, $|W_i| \gg 1$. It means that the cross section of the fusion reaction is then proportional to $1/|W_i| \ll 1$, also when the absorption is very strong. In other words, there must be a suitable value of U_{1i} in between, which makes the fusion cross section maximized at the resonance. This is the value of U_{1i} which makes $W_i = -1$ at $W_r = 0$.

This can be understood if we notice that absorption acts like damping in a resonance. The energy absorbed by a damping mechanism is proportional to the product of the damping coefficient and the square of the amplitude of the oscillation. When the damping coefficient is zero, the energy absorbed by damping mechanism is zero even if the resonance develops fully. On the other hand, when the damping coefficient is too large, the damping mechanism will kill the resonance before it is fully developed. Thus, the energy absorbed by the damping mechanism is still very small. Hence, there must be a suitable damping which makes the absorbed energy maximized. Similarly, the fusion cross section is proportional to the product of U_{1i} and the square of the amplitude of the wave function inside the nuclear well; therefore, there should be a suitable damping U_{1i} to make the fusion cross section maximized. We may call it matching damping. Consequently, one may ask the question if such a matching damping manifests itself in a nuclear resonant process.

III. EXPERIMENTAL EVIDENCE

In experiment, at the resonance energy the resonant state with the matching damping will have the largest tunneling current; hence, it should be observed first. This may be checked directly through the experimental data. The famous $d+t$ fusion process is the best candidate for this purpose, because it has a well-known resonance at the energy of 114 keV. If we assume that at this resonant energy not only $W_r = 0$, but also $W_i = -1$ to maximize the tunneling current; then, the theoretical prediction for the fusion cross section due to the S wave should be

$$\sigma_{\text{resonance}}^{(o)} = \frac{\pi}{k^2} = 4.74 \ \text{barns}. \tag{7}$$

The experimental value for the fusion cross section due to all the partial waves is

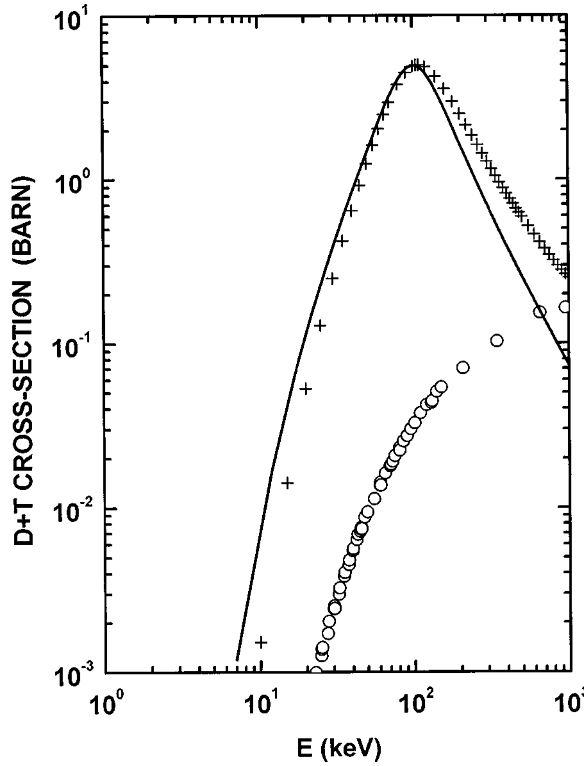

FIG. 2. Fusion cross section: experimental data for $d+t$ fusion (+); selective resonant tunneling calculation for $d+t$ fusion (solid line); and experimental data for $d+d$ fusion (o).

$$\sigma_{\text{ex}} = 4.98 \ \text{b}. \tag{8}$$

Moreover, based on the assumption of Eq. (5), we may calculate the nuclear potential under the square-well assumption. The real part (U_{1r}) and imaginary part (U_{1i}) of the nuclear potential are obtained as

$$\begin{cases} U_{1r} = -41.4 \ \text{MeV}, \\ U_{1i} = -123 \ \text{keV}. \end{cases} \tag{9}$$

Using these parameters for the nuclear well, we may further calculate the phase shift as a function of energy $[\delta_o(E)]$; hence, calculate the cross section $[\sigma_r^{(o)}(E)]$ as a function of energy E. Figure 2 shows the result of calculation. Here, the "+" and "o" denote the experimental data for $d+t$ and $d+d$, respectively. The curve is calculated in terms of Eq. (4) and nuclear well parameters of Eqs. (9) [see Appendix, Eqs. (A8) and (A9)]. The good agreement in the low-energy side is apparent. The contribution from the p wave may further improve the agreement on the high-energy side.

IV. SELECTIVE RESONANT TUNNELING

It is interesting to discuss the tunneling probability T in Eq. (4)

$$T = \frac{-4W_i}{W_r^2 + (W_i - 1)^2}. \tag{10}$$

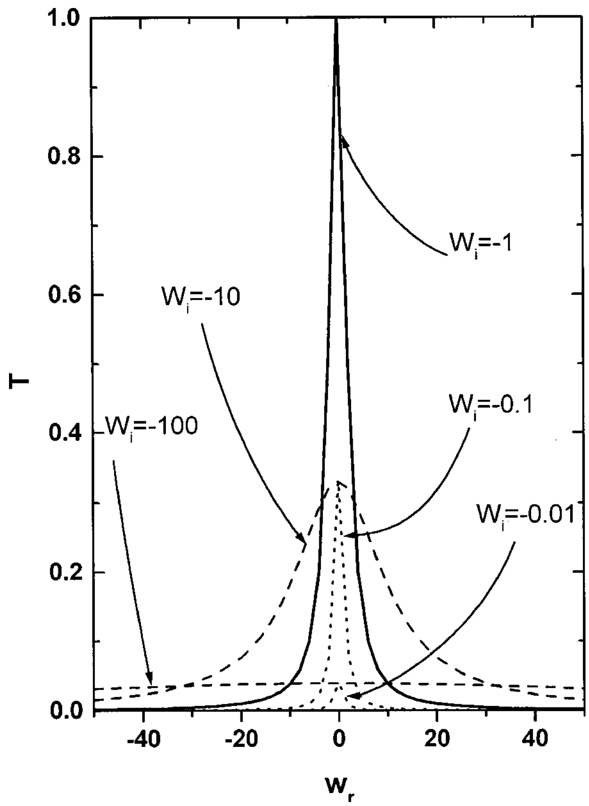

FIG. 3. Tunneling probability as a function of both W_r and W_i. The solid line is for $W_i=-1$; the dashed line is for $W_i \ll -1$; the dotted line is for $0 > W_i > -1$.

The resonant feature is clearly shown by the dependence on W_r (Fig. 3). The tunneling probability will reach its peak at $W_r=0$, and the width of this peak is determined by $|W_i-1|$. When $W_i=-1, T=1$. If W_i is greater or less than (-1); then, the peak value of T is always less than 1 (see dashed lines and dotted lines in Fig. 3). This will generate a selective feature for resonant tunneling phenomenon. As we may expect, W_r (i.e., mainly the real part of the phase shift) varies with the incident energy of the projectile. However, W_i varies with the lifetime of the state which is composed of the tunneling projectile and the target. When the incident energy is in resonance with the energy level of the composed state, the resonant tunneling happens $(W_r=0)$. However, if the lifetime of this composed state does not make $W_i=-1$; then, the tunneling probability is still low even if at this resonant energy. Thus, if there are more than one states with different lifetimes at the same energy level; then, the resonant tunneling process may generate only a few states which have the right lifetime to make $W_i \approx -1$. We may call it selective resonant tunneling. Now the question is how sharp is this selectivity. Or what is the dependence of W_i on U_{1i} (the lifetime of the composed state inside the nuclear well is $\tau \approx \hbar/|U_{1i}|$). At the resonance,

$$W_i=\theta^2 \frac{a_c}{a} \frac{z_i \sin(2z_r)-z_r \sinh(2z_i)}{2[\sin^2(z_r)+\sinh^2(z_i)]} \tag{11}$$

[see Eq. (A8)]. Here

$$\theta^2=\frac{1}{2\pi}\left[\exp\left(\frac{2\pi}{ka_c}\right)-1\right] (\theta^{-2} \quad \text{is the Gamow factor}), \tag{12}$$

$$k^2=\frac{2\mu}{\hbar^2}E, \tag{13}$$

$$z=z_r+iz_i \equiv k_{1r}a+ik_{1i}a. \tag{14}$$

a is the radius of the nuclear well. a is taken as $a=a_o(A_1^{1/3}+A_2^{1/3}), a_o=1.12\times 10^{-13}$ cm; A_1 and A_2 are the mass numbers of colliding nuclei, respectively. $a_c=\hbar^2/Z_1Z_2\mu e^2$; μ is the reduced mass of the colliding particles; Z_1e and Z_2e are the electrical charge of these particles, respectively. k_1 is the wave number inside the nuclear well, i.e.,

$$k_1^2=\frac{2\mu}{\hbar^2}[E-(U_{1r}+iU_{1i})]. \tag{15}$$

Hence, the imaginary part of k_1 is

$$k_{1i}=\frac{\mu}{k_{1r}\hbar^2}(-U_{1i}). \tag{16}$$

From expressions (11), (14), and (16), we may observe the dependence of W_i on U_{1i}. When $U_{1i}=0$ (no absorption), $z_i \equiv k_{1i}a=0$; then, $W_i=0$ and $T=0$. On the other hand, if $|U_{1i}|$ is very large and it makes $|z_i| \equiv |k_{1i}a| \gg 1$; then, $|W_i|$ rises quickly with $(k_{1i}a)$ due to the large factor θ^2 in Eq. (11). When $|W_i| \approx O(\theta^2) \gg 1, T \approx O(4/\theta^2) \ll 1$.

Thus, we can see that even if at the resonance $(W_r=0)$, the tunneling probability T is still very small if $|U_{1i}|$ is too large or too small. However, there is a suitable value for $|U_{1i}|$ to make $T=1$ at resonance. When

$$U_{1i} \approx -O\left(\frac{1}{\theta^2}\right) \tag{17}$$

then

$$|z_i| \equiv |k_{1i}a| \approx O\left(\frac{1}{\theta^2}\right) \ll 1 \tag{18}$$

and

$$W_i=\theta^2 \frac{a_c}{a} \frac{\left[\frac{1}{\theta^2}\sin(2z_r)-z_r \sinh\left(\frac{2}{\theta^2}\right)\right]}{2\left[\sin^2(z_r)+\sinh^2\left(\frac{1}{\theta^2}\right)\right]}$$

$$\approx \theta^2 \frac{a_c}{a} \frac{[\sin(2z_r)-2z_r]\left(\frac{1}{\theta^2}\right)}{2\left[\sin^2(z_r)+\left(\frac{1}{\theta^2}\right)^2\right]} \approx -1. \tag{19}$$

Hence, it is possible to make $W_i = -1$ if $|U_{1i}|$ is a small number of the order of $1/\theta^2$. We may call it the matching-damping value which is determined by Gamow factor (θ^{-2}).

When the damping (U_{1i}) deviates from this matching-damping value, the tunneling probability T will approach zero quickly due to the large factor θ^2 in front of Eq. (11). Usually, for the low-energy tunneling ($k^2 \to 0$), θ^2 is a very large number due to the exponential factor in Eq. (12). Hence, the selectivity on U_{1i} would be very sharp. Tunneling probability at resonance would vary from 0 to 1 when U_{1i} changes from 0 to $-1/\theta^2$. Tunneling probability would drop quickly from 1 to $O(1/\theta)$ when U_{1i} changes from $-1/\theta^2$ to $-1/\theta$. This is a very sharp selectivity on damping.

V. SUPPRESSION OF THE NEUTRON EMISSION IN SUB-BARRIER FUSION

The sharp selectivity in damping will suppress the neutron emission reaction in the low-energy sub-barrier fusion. This may be seen from the physical meaning of the expression

$$|z_i| \equiv |k_{1i}a| \approx O\left(\frac{1}{\theta^2}\right). \tag{20}$$

Indeed, $|k_{1i}a|$ is the ratio of the flight time to lifetime of the tunneling particle inside the nuclear well. The flight time τ_{flight} is defined as

$$\tau_{\mathrm{flight}} \approx \frac{a}{v}. \tag{21}$$

Here, v is the speed of the tunneling particle in the nuclear well; a is the size of this well. In deed τ_{flight} is of the order of the wave bouncing time in the nuclear well. In order to have a resonant tunneling the de Broglie wave of a tunneling particle should have enough bounces to build up the wave amplitude in terms of constructive interference in its lifetime. The lifetime of the tunneling particle is determined by the absorption, i.e., the imaginary part of the potential (U_{1i})

$$\tau \approx \frac{\hbar}{|U_{1i}|}. \tag{22}$$

Hence,

$$\frac{\tau_{\mathrm{flight}}}{\tau} \approx \frac{a}{v\hbar}|U_{1i}| \approx \frac{a}{\frac{k_{1r}\hbar}{\mu}\hbar}|U_{1i}| = \frac{\mu a}{k_{1r}\hbar^2}|U_{1i}| = |k_{1i}a| \equiv |z_i|\,. \tag{23}$$

This is just the definition of the z_i in Eqs. (14) and (16).

The matching damping requires

$$z_i \equiv \frac{\tau_{\mathrm{flight}}}{\tau} \approx O\left(\frac{1}{\theta^2}\right). \tag{24}$$

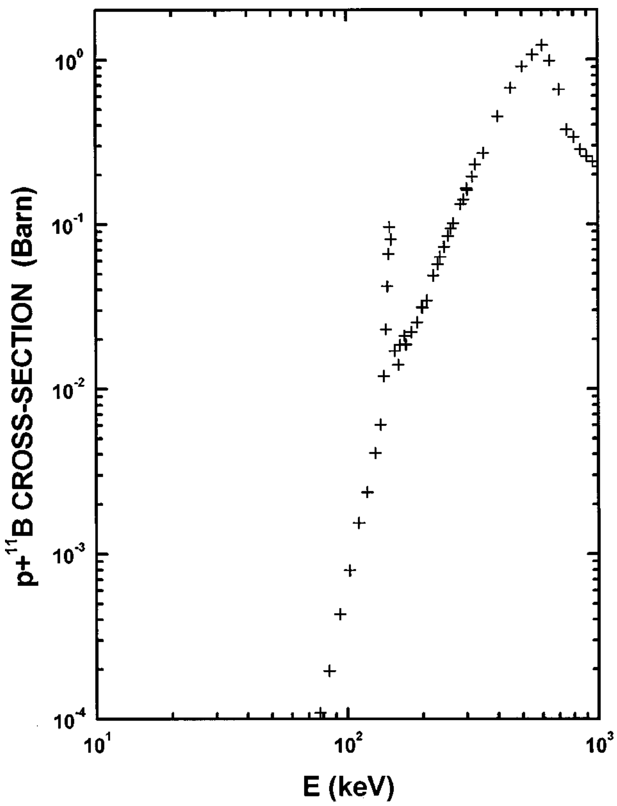

FIG. 4. $p + {}^{11}$B fusion cross section.

It implies a long lifetime τ

$$\tau \approx O(\theta^2 \tau_{\mathrm{flight}}). \tag{25}$$

Thus, the sharp selectivity in $|U_{1i}|$ is equivalent to the sharp selectivity in the lifetime of the tunneling particle (τ).

The neutron emission process is a strong interaction process controlled by the strong nuclear force. Its reaction time is of the order of the flight time inside the nuclear well, i.e., $\approx 10^{-23}$ s. Hence, if any resonant tunneling results in neutron emission, the θ^2 should *not* be very large according to Eq. (25). Or the Coulomb barrier for this resonant tunneling should not be very thick and high. Indeed, this is the case for $d+t$ fusion where a resonant tunneling happens at 114 keV with $\theta^2 < 4$.

On the other hand, if the resonant tunneling happens at a thick and high Coulomb barrier ($\theta^2 \gg 1$ for a low-energy or high-Z number); then, the sharp selectivity in damping would suppress any neutron emission reaction. For example, $p + {}^{11}$B fusion reaction is famous for its low neutron radiation and large cross section. Although it has a charge number of 5 for boron, its fusion cross section is much greater than that of $d+d$ fusion at similar energy due to the resonant tunneling. There are two resonances at $E = 148$ keV and $E = 600$ keV [6] (Fig. 4). The corresponding θ^2 values are 3.8×10^4, and 75, respectively. Hence, we may anticipate that the matching damping corresponds to a long lifetime of $3.8 \times 10^4 \tau_{\mathrm{flight}}$ or 75 τ_{flight}. They are much greater than the lifetime for neutron emission reaction ($\approx 4\tau_{\mathrm{flight}}$). Consequently, we do not observe any neutron emission in the $p+{}^{11}\mathrm{B}$ sub-barrier fusion process.

A similar argument may be applied to the $p+{}^{6}\mathrm{Li}$ and $p+{}^{13}\mathrm{C}$ sub-barrier fusion. Both of them have a very sharp resonance at low energy without any neutron emission [6,7].

Due to the narrowness of these resonances, the exact peak value for cross section is not available yet because of the difficulties in measurement. Hence, we are not able to push this qualitative evidence to the quantitative evidence. However, the cross section of $p+{}^{11}\mathrm{B}\rightarrow\alpha+{}^{8}\mathrm{Be}$ resonance at 600 keV reaches the right peak value of 1.226 b (where the selective resonant model predicts the peak value of $\sigma^{(o)}_{\mathrm{resonance}}=1.178$ b).

VI. CONCLUDING REMARKS: SELECTIVE RESONANT TUNNELING VERSUS THE COMPOUND NUCLEUS MODEL

The selective resonant tunneling model means that nuclear resonance selects not only the frequency (energy level), but also the damping (nuclear reaction). The selectivity becomes very sharp, when the resonance happens in a low energy sub-barrier tunneling. Thus, the neutron-emission reaction is suppressed in such selective resonant tunneling processes.

The compound nucleus model may not be applied to light nuclei fusion, because the penetrating particle may still remember its phase factor of the wave function, while the compound nucleus model assumes that the penetrating particle loses memory of its history [8]. In the compound nucleus model, the nuclear reaction is divided into two steps: penetrating first, then decaying. In selective resonant tunneling, the tunneling probability depends on the lifetime of decay. The tunneling process is completed in one single step. The surprisingly good agreement between the calculated cross section and experimental value for $d+t$ and $p+{}^{11}\mathrm{B}$ sub-barrier fusion is strong evidence showing that, the tunneling process is a single step process. The discovery of the nuclear halo state [9] is another strong evidence showing that even if inside the strongly interacting nuclear well region, the nucleon may still keep its own feature without losing its memory of the wave function.

The Breit-Wigner formalism for the resonant interaction requires two parameters for each resonance: the energy and the width for the resonance. However, the selective resonant tunneling model for sub-barrier fusion requires only one parameter—the energy of the resonance; the width of the resonance is then determined by the Gamow factor $[U_{1i}\approx -O(\theta^{-2})]$. When we calculated the curve for the $d+t$ cross section in Fig. 2, we did not use any input from the experiment for width; instead, we assumed $W_i=-1$ for the maximum tunneling, which is the result of the selective resonant tunneling.

Just as pointed by Balantekin [10], the fusion of two nuclei at very low energies are not only of central importance for stellar energy production and nucleosynthesis, but also provide new insights into reaction dynamics and nuclear structure.

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APPENDIX: CONNECTION BETWEEN THE COULOMB POTENTIAL AND THE SQUARE-WELL POTENTIAL

The wave function inside the nuclear well ($r<a$) is determined by two parameters, the real and the imaginary part of the nuclear potential (U_{1r} and U_{1i}). The Coulomb wave function outside the nuclear well ($r>a$) is determined by two other parameters as well: the real and the imaginary part of the phase shift $[(\delta_o)_r$ and $(\delta_o)_i]$. A pair of convenient parameters, W_r and W_i, are introduced to make a linkage between the cross section and the nuclear well. Then, it is easy to discuss the resonance and the selectivity in damping. The connection of the wave function at the boundary ($r=a$) can be expressed by the logarithmic derivative of the wave function. In the square well, the dimensionless logarithmic derivative is

$$a\frac{[\sin(k_1 r)]'}{\sin(k_1 r)}\bigg|_{r=a}=k_1 a\frac{\cos(k_1 a)}{\sin(k_1 a)}=(k_1 a)\cot(k_1 a). \tag{A1}$$

In the Coulomb field, the dimensionless logarithmic derivative has been given by Landau [5] as

$$\frac{a}{a_c}\left\{\frac{1}{\theta^2}\cot(\delta_o)+2\left[\ln\left(\frac{2a}{a_c}\right)+2C+h(ka_c)\right]\right\}, \tag{A2}$$

$$\theta^2=\frac{1}{2\pi}\left[\exp\left(\frac{2\pi}{ka_c}\right)-1\right]. \tag{A3}$$

Here, k is the wave number outside the nuclear well,

$$k^2=\frac{2\mu}{\hbar^2}E, \tag{A4}$$

a_c is the Coulomb unit of length,

$$a_c=\frac{\hbar^2}{Z_1 Z_2 \mu e^2}, \tag{A5}$$

and $C=0.577\ldots$ is Euler's constant. $h(ka_c)$ is related to the logarithmic derivative of Γ function

$$h(x)=\frac{1}{x^2}\sum_{n=1}^{\infty}\frac{1}{n(n^2+x^{-2})}-C+\ln(x). \tag{A6}$$

δ_o is the complex phase shift of the wave function due to the nuclear interaction

$$\cot(\delta_o)=W_r+iW_i. \tag{A7}$$

Having made use of Eqs. (A1), (A2), and (A7), we have

$$W_i=\theta^2 \mathrm{Im}\left[\frac{a_c}{a}(k_1 a)\cot(k_1 a)\right] = \theta^2\frac{a_c}{a}\frac{z_i\sin(2z_r)-z_r\sinh(2z_i)}{2[\sin^2(z_r)+\sinh^2(z_i)]}, \tag{A8}$$

$$W_r=\theta^2\left\{\frac{a_c}{a}\frac{z_r\sin(2z_r)+z_i\sinh(2z_i)}{2[\sin^2(z_r)+\sinh^2(z_i)]} -2\left[\ln\left(\frac{2a}{a_c}\right)+2C+h(ka_c)\right]\right\}. \tag{A9}$$

[1] S. Glasstone and R. H. Lovberg, *Controlled Thermonuclear Reactions* (Van Nostrand, Princeton, NJ, 1960), p. 16.
[2] X. Z. Li, Czech. J. Phys. **49**, 985 (1999).
[3] X. Z. Li, C. X. Li, and H. F. Huang, Fusion Technol. **36,** 324 (1999).
[4] X. Z. Li, *Current Trends in International Fusion Research: Review and Assessment,* Proceedings of the Third Symposium, edited by E. Panarella, Washington D.C., 1999, NRC Research, National Research Council of Canada, Ottawa, ON K1A 0R6 Canada (in press).
[5] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1987), p. 572.
[6] National Nuclear Data Center Web Site (http://www.nndc.bnl.gov/).
[7] W. A. Fowler, G. A. Caughlan, B. A. Zimmerman, Annu. Rev. Astron. Astrophys. **5**, 525 (1967).
[8] H. Feshbach, *Theoretical Nuclear Physics* (Wiley, New York, 1992), p. 23.
[9] K. Riisager, Rev. Mod. Phys. **66**, 1105 (1994).
[10] A. B. Balantekin and N. Takigawa, Rev. Mod. Phys. **70**, 77 (1998).